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Exact solution of the lock step model of vicious walkers

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Abstract. The lock step model of vicious walkers on a one-dimensional lattice allows each walker at the tick of a clock to move either one lattice site to the left or one lattice site to the right. The only restriction is that no two walkers may arrive at the same lattice site or pass one another. In periodic boundary conditions the partition function and correlation function for this model are calculated exactly. Taking the continuum limit gives an exactly solvable model of vicious walkers undergoing Brownian motion.

1. Introduction

1.1. *Domain walls in two-dimensional solids*

Domain walls refer to the boundary between two different allowed states of a statistical system. The usefulness of the concept in providing quantitative predictions regarding the phases of two-dimensional model systems was first revealed in the work of Pokrovsky and Talopov [1]. These authors were studying a monolayer of atoms adsorbed on a surface which forms a two-dimensional lattice of possible adsorption sites.

The simplest situation to treat theoretically is that in which the periodic potential experienced by the adsorbed atoms is very much stronger in one direction, the y direction say, than the x direction. Furthermore, suppose that the natural lattice spacing of the adsorbate in the x direction is slightly incommensurate to that of the underlying lattice. In the y direction the spacings are assumed commensurate. Then a phase in which the lattice spacing of the adsorbate is commensurate with the underlying lattice in both the x and y directions will develop at low temperature (see figure 1). As the temperature (or density, pressure, etc.) is increased this will give way to an incommensurate phase.

As a phenomological theory close to this phase transition, it has been proposed [2] that the incommensurate phase consists of a sequence of commensurate phases each separated by domain walls of mean spacing v. The spacing v is assumed large in comparison to the width of a wall. The important feature of this theory with regard to a transition to a commensurate phase is the existence of an interaction between the domain walls. The interaction is simply taken as a 'hard core' so that the walls cannot intersect each other, and the domain walls themselves are approximated as single piecewise straight lines fixed at both ends of the system in the y direction (see figure 2). Each wall has a Boltzmann weighting associated with its precise shape, but otherwise no other interactions are considered. A similar model system arose in the work of Villain and Bak [3] on the two-dimensional ANNNI model.

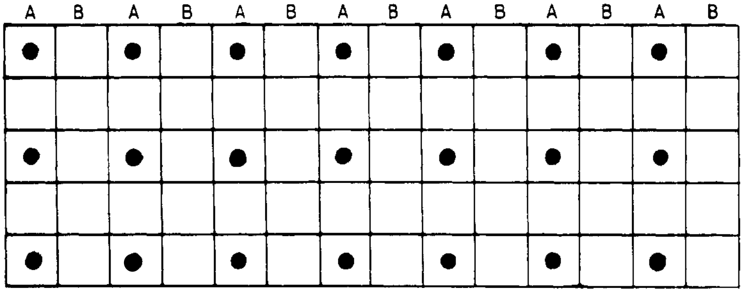

Figure 1. One of two possible commensurate phases when the natural lattice spacing of the adsorbate is approximately twice the lattice spacing of the underlying lattice. Dividing the lattice into two sublattices in the x direction, A and B as indicated, this can be labelled the A phase. The other commensurate phase, the B phase, has each adsorbate atom shifted across one lattice spacing.

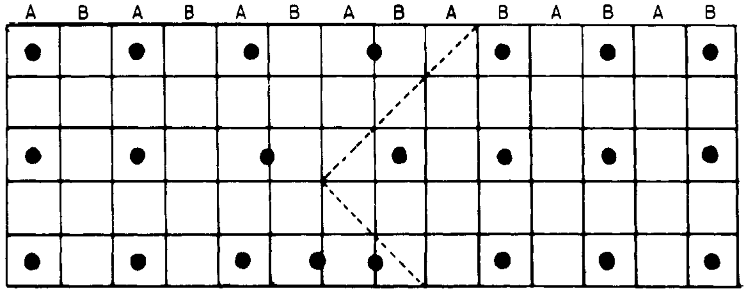

Figure 2. A typical configuration in the phenomological theory close to the commensurate–incommensurate transition. The A phase and the B phase are separated by a thin layer (domain wall) in which the lattice spacing of the adsorbate is incommensurate with that of the underlying lattice. The domain wall is approximated by a single piecewise straight line.

1.2. Vicious random walkers—the lock step model

A precise model of non-intersecting domain walls can be formulated as follows [4]. Consider a square lattice with $M \times n$ sites wrapped around a cylinder in the x direction. Along the bottom (circular) row mark in N dots at N lattice sites, all equally spaced with spacing ν (assume M/N is an even integer). Also, on the top row mark in N dots at precisely the same location within the row as on the bottom row. Construct a partition function consisting of a sum over all possible paths between dots in the bottom and top rows. Each path must start at the bottom row, and move one row at a time to the top row according to the rules:

(1) no path may go backwards or intersect any other path;

(2) each dot in successive rows must either move one column to the left with weighting w_{-1} or one column to the right with weighting w_1 (see figure 3).

This is clearly a problem in random walks, with the dots in each successive row representing the position of the walkers at each successive time interval. Walkers whose paths cannot intersect have been termed vicious by Fisher [4] and this particular model has been called the lock step model of vicious walkers.

1.3. Aim and summary of the paper

The primary objective of this paper is to complete an exact treatment of the lock step

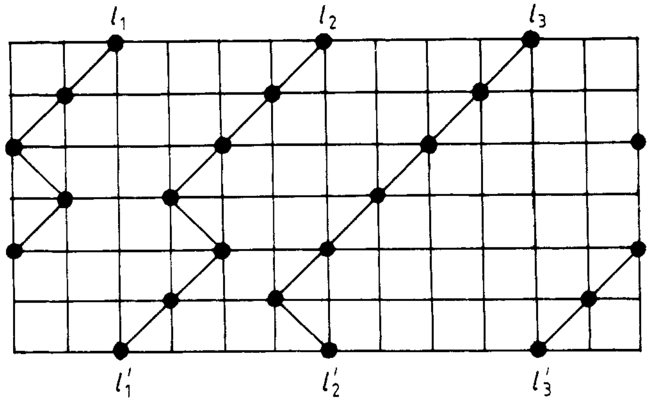

Figure 3. A typical allowed path for the lock step model with periodic boundary conditions. The lattice edges on the left and the right are to be identified. As illustrated in this particular path, the final ordering of the walkers may be cyclically permuted with respect to the initial positions.

model begun by Fisher [4] who showed that the partition function for a generalised version of the model presented above (see section 2.1 for its precise definition) in free boundary conditions can be expressed as a determinant. The first topic addressed here is the derivation of the analogue of this result for periodic boundary conditions using a difference equation. A very simple formula for the partition function of the lock step model as presented in section 1.2 is then obtained (equation (2.28)). This allows the variation of ν, the mean spacing between walls, to be calculated as a function of the weights w_{-1} and w_1. The well known [2] commensurate–incommensurate phase transition is exhibited.

In section 3, exact evaluations of one- and two-point correlation functions are performed, and presented for both the strip and two-dimensional system. The known [5, 6] critical behaviour of the correlation length is recovered. In the final section the continuum limit is taken, whereby the discrete vicious walkers become a model of non-intersecting Brownian motion particles.

Before beginning the study, it is appropriate to remark that similar discrete walker problems have previously been studied by Villain and Bak [3] and Shultz [5]. However, neither of these works considered the lock step model and consequently were restricted to an analysis valid only in the large lattice size limit. Also, as remarked above, in free boundary conditions the determinant formula has previously been obtained by Fisher [4] and Huse and Fisher [6]. Their interest was in calculating the long-time behaviour of the partition function for a fixed number of walkers, which is related to the large-separation behaviour of correlation functions in Ising and higher-state two-dimensional spin models. Here our interest is in the limit in which both the number of steps and the number of walkers become infinite.

2. The partition function for the lock step model

2.1. *A more general version of the lock step model*

Let the lattice sites along any row be labelled $1, 2, \ldots, M$ according to their position on the row. Recall that the lattice has been wrapped around a cylinder so that the Mth site of any row is next to the first. Generalise the model of section 1.2 to allow the N walkers to initially be at the lattice sites $\ell'_1, \ell'_2, \ldots, \ell'_N$. The sites ℓ'_j ($j = 1, \ldots, N$) are arbitrary except that they must all be different and form part of the even-numbered sublattice. They may be ordered so that

$$1 \leq \ell'_1 < \ell'_2 < \ldots < \ell'_N \leq M. \tag{2.1}$$

Let the final position of the walkers be at $\ell_1, \ell_2, \ldots, \ell_N$ on the nth row. These positions may be ordered so that

$$\ell_1 < \ell_2 \ldots < \ell_N < \ell_1 \tag{2.2}$$

where here the inequality $a < b$ means a is to the left of b when facing towards the centre of the cylinder. The walks between the lattice sites (2.1) on the first row and the lattice sites (2.2) on the nth row are again subject to the rules (1) and (2) of section 1.2. It is convenient to think of ℓ_j as the final position of the jth walker. However, ℓ'_j is then not necessarily the initial position of the jth walker. In general, the initial position of the jth walker may be any one of the lattice sites $\ell'_1, \ldots, \ell'_N$ (see figure 3).

The partition function Z of this model can be written

$$Z(\ell'_1, \ldots, \ell'_N \mid \ell_1, \ldots, \ell_N; n) = \sum_{\rho \in \wp} \prod_{j=1}^{N} W_n(\ell'_k \mid \ell_j). \tag{2.3}$$

Here $\wp$ denotes the set of all allowed paths, according to the rules of section 1.2. Any one member ρ of this set consists of N paths. Each of the N paths is weighted according to rule (2) of sections 1, 2; the jth path of $\wp$ having total weight $W_n(\ell'_k \mid \ell_j)$.

2.2. *A multi-variable difference equation*

According to rule (2) of section 1.2, the partition function with $n+1$ rows is related to a sum over partition functions with n rows. The equation is most compactly written by denoting

$$Z(\ell'_1, \ldots, \ell'_N \mid \ell_1, \ldots, \ell_N; n) = Q(\ell_1; n) Q(\ell_2; n) \ldots Q(\ell_N; n). \tag{2.4}$$

(Note well: it is not possible to factorise Z as such; this is merely a convenient notation.) Then

$$Z(\ell'_1, \ldots, \ell'_N \mid \ell_1, \ldots, \ell_N; n+1) = \prod_{j=1}^{N} [w_{-1} Q(\ell_j - 1; n) + w_1 Q(\ell_j + 1; n)]. \tag{2.5}$$

This multi-variable difference equation is to be solved subject to the initial condition

$$Z(\ell'_1, \ldots, \ell'_N \mid \ell_1, \ldots, \ell_N; 0) = \prod_{j=1}^{N} \prod_{k=1}^{N} \delta_{\ell'_j, \ell_{j+k}} \tag{2.6}$$

which, when thinking of ℓ_k as representing the final coordinates of the walkers, says that the initial coordinates may be any cyclic combination of $\ell'_1, \ldots, \ell'_N$ in order. Here

$$\ell_{j+k} := \ell_{j+k-N} \qquad \text{if } j + k > N. \tag{2.7}$$

Since the lattice has been wrapped around a cylinder, it is necessary that the solution of (2.5) satisfies the periodicity condition

$$Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_j+M,\ldots,\ell_N;n) = Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_N;n) \quad (2.8)$$

for each $j = 1,2,\ldots,N$.

The non-crossing condition for the paths must also be satisfied. A crucial feature of the solvability of the lock step model is that for the paths to cross, the walkers must first arrive at the same lattice site. Thus the non-crossing condition can be replaced by the non-intersecting condition

$$Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_j,\ldots\ell_k,\ldots,\ell_N;n) = 0 \qquad \text{if } \ell_j = \ell_k \quad (2.9)$$

for any $j \neq k = 1,2,\ldots,N$.

The conditions (2.1), (2.2) (with $<$ replaced by $\leq$), and (2.6)–(2.8) specify a unique solution of (2.5). For N odd, the required solution is given by the following result.

Theorem 2.1. Let N be odd. Then the partition function of the above-specified lock step model is given by

$$Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_N;n) = \det[Q_n^{(0)}(\ell'_j \mid \ell_k)]_{j,k=1,\ldots,N} \quad (2.10)$$

where

$$Q_n^{(0)}(\ell'_j \mid \ell_k) = \frac{1}{M}\sum_{\alpha=0}^{M-1} \mathrm{e}^{-2\pi\mathrm{i}(\ell_k-\ell'_j)\alpha/M}[\phi(2\pi\alpha/M)]^n \quad (2.11)$$

and

$$\phi(\theta) = w_{-1}\mathrm{e}^{-\mathrm{i}\theta} + w_1\mathrm{e}^{\mathrm{i}\theta}. \quad (2.12)$$

Remark. $Q_n^{(0)}(\ell'_j \mid \ell_k)$ is the partition function of a single walker on a cylinder, starting at site ℓ'_j on the first row and finishing at site ℓ_k on the nth row.

Proof. Since $Q_n^{(0)}(\ell'_j \mid \ell_k)$ is the partition function of a single walker it satisfies the difference equation

$$Q_{n+1}^{(0)}(\ell'_j \mid \ell_k) = w_{-1}Q_n^{(0)}(\ell'_j \mid \ell_k - 1) + w_1 Q_n^{(0)}(\ell'_j \mid \ell_k + 1). \quad (2.13)$$

In (2.10), with n replaced by $n+1$, use of (2.13) in the first row gives

$$\begin{aligned} Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_N;n+1) &\\ = \det&\begin{bmatrix} w_{-1}Q_n^{(0)}(\ell'_1 \mid \ell_k - 1) + w_1 Q_n^{(0)}(\ell'_1 \mid \ell_k = 1) \\ Q_n^{(0)}(\ell'_j \mid \ell_k) \end{bmatrix}_{\substack{j=2,\ldots,N\\k=1,\ldots,N}} \\ = w_{-1}\det&\begin{bmatrix} Q_n^{(0)}(\ell'_1 \mid \ell_k - 1) \\ Q_{n+1}^{(0)}(\ell'_j \mid \ell_k) \end{bmatrix}_{\substack{j=2,\ldots,N\\k=1,\ldots,N}} \\ + w_1\det&\begin{bmatrix} Q_n^{(0)}(\ell'_1 \mid \ell_k + 1) \\ Q_{n+1}^{(0)}(\ell'_j \mid \ell_k) \end{bmatrix}_{\substack{j=2,\ldots,N\\k=1,\ldots,N}} \end{aligned} \quad (2.14)$$

where the first entry in each determinant holds true for the first row only, while the remaining $N-1$ rows are specified by the second term. Now apply the recurrence (2.13) and the determinant identity used to obtain the second line in (2.14) to each of the rows $2,3,\ldots,N$ in order, to each determinant in (2.14) and to the new determinants introduced in this procedure. A total of 2^N terms results. If the notation (2.4) is introduced, it is clear that these terms can be factorised according to the right-hand side of (2.5). Thus (2.10) satisfies the difference equation (2.5).

Now

$$Q_0^{(0)}(\ell'_j \mid \ell_k) = \delta_{\ell'_j,\ell_k}. \tag{2.15}$$

Furthermore $\{\ell'_j\}$ and $\{\ell_k\}$ are ordered according to (2.1) and (2.2) respectively. Thus, unless $\{\ell'_j\} = \{\ell_k\}$,

$$Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_N;0) = 0. \tag{2.16}$$

If $\{\ell'_j\} = \{\ell_k\}$, the partition function with $n=0$ as given by (2.10) is the determinant of a cyclic matrix with all entries in the top row 0 except for a single entry of 1. Since N is odd, the order of the matrix is odd, so such a cyclic matrix can be obtained from the identity by an even number of row interchanges. Hence in this case

$$Z(\ell'_1,\ldots,\ell'_N \mid \ell_1,\ldots,\ell_N;0) = 1 \tag{2.17}$$

and so the initial condition (2.6) is satisfied.

The periodicity condition (2.8) follows immediately from the periodicity

$$Q_n^{(0)}(\ell'_j \mid \ell_k + M) = Q_n^{(0)}(\ell'_j \mid \ell_k) \tag{2.18}$$

and the non-intersecting condition (2.9) is an immediate consequence of the vanishing property of determinants whenever two rows are equal. The theorem thus follows.

It remains an open problem to obtain an identity analogous to (2.10) for N even.

2.3. Evaluation of the partition function for equally spaced initial and final positions

Consider again the lock step model as formulated in section 1.2. This corresponds to the model as formulated in the previous subsection with

$$\ell'_j = \ell_j = jv \qquad j = 1,2,\ldots,N \tag{2.19}$$

where

$$v = M/N \tag{2.20}$$

is the average spacing between walkers, and is assumed to be an even integer. Let the partition function of this model be denoted $Z_n^{(N)}$. Then from (2.10) and (2.12), provided N is odd,

$$Z_n^{(N)} = \det\left[Q_n^{(0)}(j \mid k)\right]_{j,k=1,\ldots,N} \tag{2.21}$$

where

$$Q_n^{(0)}(j \mid k) = \frac{1}{M} \sum_{\alpha=0}^{M-1} e^{-2\pi i(k-j)\alpha/N} \left[\phi(2\pi\alpha/M)\right]^n$$
$$:= F_{k-j}. \tag{2.22}$$

We note

$$F_{k-j} = F_{j-k} = F_{k-j+N}. \tag{2.23}$$

Thus

$$Z_n^{(N)} = \det \begin{bmatrix} F_0 & F_1 & \cdots & F_{N-1} \\ F_{N-1} & F_0 & \cdots & F_{N-2} \\ \vdots & \vdots & & \vdots \\ F_1 & F_2 & \cdots & F_0 \end{bmatrix}_{N\times N} \tag{2.24}$$

and so in this case the partition function is the determinant of a cyclic matrix. Hence, by the well known formula for the evaluation of such a determinant [7],

$$\begin{aligned} Z_n^{(N)} &= \prod_{\ell=0}^{N-1} \sum_{a=0}^{N-1} e^{2\pi i a\ell/N} F_a \\ &= \prod_{\ell=0}^{N-1} \frac{1}{M} \sum_{a=0}^{N-1} e^{2\pi i a\ell/N} \sum_{\alpha=0}^{M-1} e^{-2\pi i a\alpha/N} [\phi(2\pi\alpha/M)]^n. \end{aligned} \tag{2.25}$$

Equation (2.25) can be simplified by noting that

$$\sum_{a=0}^{N-1} e^{2\pi i a(\ell-\alpha)/N} = N\delta_{\ell-\alpha,bN} \tag{2.26}$$

for any integer b. Since $0 \le a \le N-1$, $0 \le \alpha \le M-1$ and $M = vN$, the non-zero cases in (2.26) occur when

$$b = 0, -1, -2, \ldots, -(v-1). \tag{2.27}$$

Hence

$$\begin{aligned} Z_n^{(N)} &= \prod_{\ell=0}^{N-1} \frac{1}{v} \sum_{b=0}^{v-1} [\phi(2\pi(\ell/M + b/v))]^n \\ &= \prod_{\ell=0}^{N-1} \frac{(1+(-1)^n)}{v} \sum_{b=0}^{v/2-1} [\phi(2\pi(\ell/M + b/v))]^n \end{aligned} \tag{2.28}$$

where the second line follows from the first by the antiperiodicity property of ϕ as defined by (2.12). Note that $Z_n^{(N)}$ vanishes if n is odd, as it must from the definition of the model.

As an illustration of the simplicity and content of (2.28), take $n = M = 12$ and $N = 3$. With $w_{-1} = w_1 = 1$, $Z_{12}^{(3)}$ counts the number of non-intersecting paths, formed according to the rules of section 1.2, between the sites $\ell'_1 = 4, \ell'_2 = 8$ and $\ell'_3 = 12$ on the first row and the same sites on the top row of a 12×12 lattice wrapped around a cylinder. From (2.28) and (2.12)

$$Z_{12}^{(3)} = 2^9(1+3^6)^2 = 272\,844\,800. \tag{2.29}$$

2.4. Free energy of the lock step model

Let

$$f_n(v) = -\lim_{M\to\infty} \frac{1}{M} \log Z_n^{(N)} \tag{2.30}$$

denote the dimensionless free energy per unit length on a strip-shaped lattice of infinite length and width n rows. The average distance between walkers v is to be held fixed in the limiting procedure. From (2.28) we see that taking the logarithm gives an expression which is just the Riemann sum approximation to an integral. Hence, for n even

$$f_n(v) = \frac{1}{v}\left[\log(v/2) - \int_0^1 \log\left(\sum_{b=0}^{v/2-1} [\phi(2\pi(t+b)/v)]^n\right)\mathrm{d}t\right]. \tag{2.31}$$

Now suppose $w_1 = w_{-1}$ so that

$$\phi(\theta) = 2w_1 \cos\theta \tag{2.32}$$

and consider (2.31) in the large number of steps n limit. Denoting the free energy per lattice site in the two-dimensional thermodynamic limit as $F(v)$, we then have

$$F(v) = -\frac{1}{v}\log(2w_1) - \int_0^{1/v} \log|\cos 2\pi t|\mathrm{d}t. \tag{2.33}$$

2.5. Phase transition in the domain wall model

The model of domain walls by vicious walkers requires that the distance between walls be a variable. For a given Boltzmann weight w_1, the spacing v which minimises the free energy $F(v)$ is the required choice. Regarding v as a continuous variable, from (2.33) the minimum occurs when

$$2w_1 = [\cos(2\pi/v)]^{-1}. \tag{2.34}$$

It follows that for $2w_1 \leq 1$ the value of v which minimises the free energy v is $v \longrightarrow \infty$. In the model of section 1.1, this corresponds to a commensurate phase.

For $2w_1 > 1$ there are a series of incommensurate phases characterised by the mean spacing v (which must be an even integer) and separated by first-order transitions. As $2w_1 \longrightarrow 1^+$, the transitions becomes quasicontinuous and we see from (2.34) the behaviour

$$v \sim \frac{\pi}{(w_1 - \frac{1}{2})^{1/2}}. \tag{2.35}$$

The exponent $\frac{1}{2}$ is well known [1,8].

Minimisation of the strip-system free energy (2.31) with respect to v also yields a commensurate phase for $2w_1 \leq 1$, and a series of incommensurate phases with a finite value of v depending on w_1 and n for $2w_1 > 1$. However, there is no divergence of v as the transition is approached from the incommensurate phase. For $v > n$, the partition function factorises into that for non-interacting walkers, and the minimum free energy in these cases occurs when $v = n + 2$. This is a bound on v in the incommensurate phase of the strip system.

3. Correlations for the lock step model

3.1. Transformation of the partition function of the generalised lock step model

In order to calculate correlation functions, it is convenient to first transform the partition function (2.10) in the case when the walkers are initially equally spaced so that $\ell'_j = vj$ for each $j = 1, 2, \ldots, N$. The transformation is accomplished by multiplying (2.10) in this case by unity in the form of

$$i^{N(N-1)/2} N^{-N/2} \det[e^{-2\pi i k m/N}]_{\substack{k=1,2,\ldots,N \\ m=-(N-1)/2,\ldots,(N-1)/2}} \tag{3.1}$$

(see e.g. [7] for the derivation of this result). Thus, after straightforward manipulation, we have

$$\begin{aligned} Z(v, 2v, \ldots, Nv \mid \ell_1, \ldots, \ell_N; n) &= i^{N(N-1)/2} N^{-N/2} v^{-N} \\ &\quad \times \det\left\{ \sum_{b=0}^{v-1} e^{-2\pi i \ell_j (k+bN)/M} \left[\phi\left(\frac{2\pi}{M}(k+bN) \right) \right]^n \right\}_{\substack{j=1,2,\ldots,N \\ k=-(N-1)/2,\ldots,(N-1)/2}} \end{aligned} \tag{3.2}$$

3.2. Calculation of the one-point function

The one-point function we shall calculate is the density ${}_n\rho(\ell^*, n^*)$ that a vicious walker will arrive at the lattice site ℓ^* in n^* steps and then arrive at the final lattice site in a total of n steps. The final and initial configurations are taken as being equally spaced and the steps are formed according to rules (1) and (2) of section 1.2. Explicitly,

$$\begin{aligned} {}_n\rho(\ell^*, n^*) &= (Z_n^{(N)})^{-1} \frac{\delta}{\delta A(\ell^*)} \sum_R \prod_{j=1}^{N} (1 + A(\ell_j)) \\ &\quad \times Z(v, \ldots, Nv \mid \ell_1, \ldots, \ell_N; n^*) Z(\ell_1, \ldots, \ell_N \mid v, \ldots, Nv; n - n^*) \big|_{A=0} \end{aligned} \tag{3.3}$$

where R denotes the region (2.2) and $\delta/\delta A$ denotes functional differentiation. Since the summand in (3.3) is symmetrical in the ℓ_j we can sum from 1 to M in each variable provided we divide by $N!$.

Now substitute the identity (3.2) in (3.3) and expand out the determinant using the defining formula

$$\det[X_{j,k}]_{j,k=1,2,\ldots,N} = \sum_{P=1}^{N!} \varepsilon(P) \prod_{\ell=1}^{N} X_{\ell, P(\ell)}. \tag{3.4}$$

An expression of the form

$$\sum_{P=1}^{N!} \sum_{Q=1}^{N!} \varepsilon(P)\varepsilon(Q) Y_{P(\ell),Q(\ell)} \tag{3.5}$$

results, which is simply

$$N! \det[Y_{j,k}]_{j,k=1,2,\ldots,N}. \tag{3.6}$$

This procedure yields the identity

$$
{}_n\rho(\ell^*, n^*) = (Z_n^{(N)} N^N v^{2N})^{-1} \frac{\delta}{\delta A(\ell^*)} \times \det \left[\sum_{\ell=1}^{M} \sum_{b=1}^{v-1} \sum_{c=1}^{v-1} (1 + A(\ell)) f_{k,k'}(\ell, b, c) \right]_{k,k'=-(N-1)/2,\ldots,(N-1)/2} \tag{3.7}
$$

where

$$
f_{k,k'}(\ell, b, c) = \exp[-2\pi i \ell (k - k' + (b-c)N)/M] \times \left[\phi\left(\frac{2\pi}{M}(k + bN)\right)\right]^{n^*} \left[\phi\left(\frac{2\pi}{M}(k' + cN)\right)\right]^{n-n^*}. \tag{3.8}
$$

The functional differentiation can be performed row-by-row in the determinant and thus only one row, the pth, say, ($p = -(N-1)/2, \ldots, (N-1)/2$), is affected by the differentiation. Setting $A(\ell) = 0$ in the other rows gives that the only non-zero term occurs when $k = k'$ and has the value

$$
M \left[\phi\left(\frac{2\pi}{M}(k + bN)\right)\right]^{n}. \tag{3.9}
$$

The only term in the pth row which contributes to the value of the determinant is therefore also on the diagonal. Recalling (2.28) we thus have the evaluation

$$
{}_n\rho(\ell^*, n^*) = \frac{1}{M} \sum_{\alpha=-(N-1)/2}^{(N-1)/2} \sum_{b_1=0}^{v-1} \sum_{b_1'=0}^{v-1} e^{2\pi i \ell^* (b_1 - b_1')/v} \left[\phi\left(\frac{2\pi}{M}(\alpha + b_1 N)\right)\right]^{n^*} \times \left[\phi\left(\frac{2\pi}{M}(\alpha + b_1' N)\right)\right]^{n-n^*} \left\{ \sum_{b=0}^{v-1} \left[\phi\left(\frac{2\pi}{M}(\alpha + bN)\right)\right]^{n} \right\}^{-1}. \tag{3.10}
$$

Taking the two-dimensional thermodynamic limit $M, n \to \infty$ shows that the resulting density $\rho(\ell^*, n^*)$ is periodic in ℓ^* with period v. The leading behaviour for n^* large in the symmetric case (2.32) is

$$
\rho(\ell^*, n^*) = \left(1 + (-1)^{\ell^* + n^*}\right) \left\{ \frac{1}{v} + 2\cos(2\pi \ell^*/v) \int_{-1/2v}^{1/2v} \frac{\cos^{n^*}[2\pi(t + 1/v)]}{\cos^{n^*}(2\pi t)} \mathrm{d}t \right\}. \tag{3.11}
$$

3.3. The two-point correlation

The two-point correlation ${}_n\rho^{\mathrm{T}}((\ell_a, n_1), \ell_b, n_2))$ is defined as

$$
{}_n\rho^{\mathrm{T}}((\ell_a, n_1), (\ell_b, n_2)) = {}_n\rho((\ell_a, n_1), (\ell_b, n_2)) - {}_n\rho(\ell_a, n_1)\, {}_n\rho(\ell_b, n_2) \tag{3.12}
$$

where ${}_n\rho(\ell, n)$ is given by (3.3). The quantity ${}_n\rho((\ell_a, n_1), (\ell_b, n_2))$ denotes the distribution function for a walker arriving at site ℓ_a in n_1 steps and a walker arriving at site ℓ_b in

n_2 steps, and then all walkers arriving at an equally spaced configuration in n steps. Explicitly

$$
\begin{aligned}
{}_n\rho((\ell_a,n_1),(\ell_b,n_1)) &= (Z_n^{(N)}(N!)^2)^{-1}\frac{\delta^2}{\delta A(\ell_a)\delta B(\ell_b)} \\
&\times \sum_{\ell_1,\ldots,\ell_N=1}^{M}\ \sum_{\ell'_1,\ldots,\ell'_N=1}^{M}\ \prod_{j=1}^{N}(1+A(\ell_j))(1+B(\ell'_j)) \\
&\times Z(\nu,\ldots,N\nu \mid \ell_1,\ldots,\ell_N;n_1)Z(\ell_1,\ldots,\ell_N \mid \ell'_1,\ldots,\ell'_N;n_2-n_1) \\
&\times Z(\ell'_1,\ldots,\ell'_N \mid \nu,\ldots,N\nu;n-n_2)\,|_{A=B=0}\,.
\end{aligned}
\tag{3.13}
$$

For the first and third partition functions in (3.13) substitute the identity (3.2) while for the second substitute (2.10). Manipulation of the determinants as sketched in section 3.2 then shows

$$
{}_n\rho((\ell_a,n_1),(\ell_b,n_2)) = (Z_n^{(N)}N\nu^2)^{-N}\frac{\delta^2}{\delta A(\ell_a)\delta B(\ell_b)}
$$

$$
\times \det\left[\sum_{\ell,\ell',\alpha=1}^{M}\ \sum_{b,c=0}^{\nu=1}(1+A(\ell))(1+B(\ell'))g_{k,k'}(\ell,\ell',\alpha,b,c)\right]_{k,k'=-(N-1)/2,\ldots,(N-1)/2} \tag{3.14}
$$

where

$$
\begin{aligned}
g_{k,k'}(\ell,\ell',\alpha,b,c) &= \exp[2\pi i\ell(\alpha-k-bN)/M - 2\pi i\ell'(\alpha-k'-cN)/M] \\
&\times \left[\phi\left(\frac{2\pi}{M}(k+bN)\right)\right]^{n_1}\left[\phi\left(\frac{2\pi}{M}(k'+cN)\right)\right]^{n-n_2}\left[\phi\left(\frac{2\pi\alpha}{M}\right)\right]^{n_2-n_1}.
\end{aligned}
\tag{3.15}
$$

Performing the functional differentiation affects at most two rows of the determinant in (3.14). Again, with $A = B = 0$ in the remaining rows, the only non-zero term is the diagonal with the value (3.9). In the thermodynamic limit with $M,N \to \infty, M/N = \nu =$ constant and $n_1,n_2,n \to \infty$, $n_2 - n_1 = n =$ constant, this procedure yields the exact evaluation for the two-point correlation

$$
\begin{aligned}
\rho^{\mathrm{T}}(\ell_a-\ell_b;n^*) &= 2\left(\int_{-1/2\nu}^{1/2\nu}\frac{e^{-2\pi i(\ell_a-\ell_b)t}}{[\phi(2\pi t)]^{n^*}}dt\right)\left(\int_0^1 e^{2\pi i(\ell_a-\ell_b)s}[\phi(2\pi s)]^{n^*}ds\right) \\
&\quad -4\left(\int_{-1/2\nu}^{1/2\nu}\frac{e^{2\pi i(\ell_a-\ell_b)t}}{[\phi(2\pi t)]^{n^*}}dt\right)\left(\int_{-1/2\nu}^{1/2\nu} e^{-2\pi i(\ell_a-\ell_b)s}[\phi(2\pi s)]^{n^*}ds\right).
\end{aligned}
\tag{3.16}
$$

Equation (3.16) assumes the parities of the number of steps n^* and lattice sites ℓ_a and ℓ_b are such that paths to and from the initial and final configurations to these sites are permissable (otherwise $\rho^{\mathrm{T}}(\ell_a-\ell_b;n^*)$ is zero).

3.4. Asymptotic behaviour of the correlations

Let us first consider the transverse correlation $\rho^{\mathrm{T}}(\ell_a-\ell_b;0)$. From (3.16), for $\ell_a \neq \ell_b$,

$$
\rho^{\mathrm{T}}(\ell_a-\ell_b;0) = -\frac{4\sin^2[\pi(\ell_a-\ell_b)/\nu]}{\pi^2(\ell_a-\ell_b)^2} \tag{3.17}
$$

which is the same functional form as the two-particle correlations for free fermions on a line or the log-potential one-component plasma on a line at the reduced coupling $\Gamma = 2$ [9]. The correlation (3.17) decays algebraically for all ν, so the assumed functional form [10] $e^{-|\ell_a-\ell_b|/\xi}$ for the definition of the exponent $\nu_\perp$ (not to be confused with the mean-spaced ν) does not hold. However, from (3.17) the separation $|\ \ell_a - \ell_b\ |$ can always be made to occur in the ratio $\pi\ |\ \ell_a - \ell_b\ |\ /\nu$ so it seems reasonable to take $\xi = \nu$. Near the commensurate–incommensurate transition ν behaves as given by (2.35) so

$$\nu_\perp = 1/2 \tag{3.18}$$

as has been previously deduced [5, 6].

The parallel correlation occurs when $\ell_a - \ell_b = 0$. Considering the symmetrical case $w_1 = w_{-1}$, from (2.32) and (3.16) we see

$$\rho^{\mathrm{T}}(0; n^*) = 16 \int_0^{1/2\nu} (\cos 2\pi t)^{-n^*} \mathrm{d}t \int_{1/2\nu}^{1/4} (\cos 2\pi s)^{n^*} \mathrm{d}s. \tag{3.19}$$

For n^* large, asymptotic expansions of the integrals in (3.19) can be obtained by expanding both integrals about $1/2\nu$. If ν is also large, this gives

$$\rho^{\mathrm{T}}(0; n^*) \sim 4\left(\frac{\nu}{\pi^2 n^*}\right)^2 \left[1 - \exp\left(\frac{-\pi^2}{n^* \nu^2}\right)\right]. \tag{3.20}$$

Again this functional form does not agree with that assumed in the definition of the correlation length ξ. However, it is always possible to choose n^* to occur in the ratio $\pi^2 n^*/\nu^2$ so it seems reasonable to choose $\xi = \nu^2/\pi^2$. This gives the exponent as

$$\nu_\parallel = 2\nu_\perp = 1 \tag{3.21}$$

in agreement with previous work [5, 6].

4. Vicious walkers undergoing Brownian motion

It was noted by Fisher [4] that the continuum limit of the lock step model could be taken to obtain a model of vicious walkers undergoing Brownian motion. The continuum limit can be illustrated by considering the partition function (2.11) for a single discrete walker. It is necessary to define a length scale, τ say, which is the distance between nearest-neighbour lattice sites, and to redefine the partition function to include $1/\tau$ as a factor. Doing this, using the periodicity of the summand and considering the symmetric case (2.32), (2.11) becomes

$$Q_n^{(0)}(\ell'\tau, \ell\tau) = \frac{(2w_1)^n}{M\tau} \sum_{\alpha=-(M-1)/2}^{(M-1)/2} e^{-2\pi i(\ell-\ell')\alpha/M} \left(\frac{\cos(2\pi\alpha)}{M}\right)^n. \tag{4.1}$$

Following Fisher [4], define

$$2w_1 = e^{-\sigma'} \tag{4.2}$$

and let b^2 denote the mean-square displacement. For the lock step model

$$b^2 = \tau^2. \tag{4.3}$$

The continuum limit is then

$$\begin{array}{lllll} M \to \infty & \ell \to \infty & n \to \infty & \tau \to 0 & b \to 0 \\ \tau\ell \to x & \tau M \to L & b^2 n \to Dt & \sigma' n \to \sigma & \end{array} \tag{4.4}$$

where x, L, D, σ and t are all finite. Since for n and M large

$$\left(\frac{\cos(2\pi\alpha)}{M}\right)^n \sim \mathrm{e}^{-2\pi^2\alpha^2 Dt/L^2} \tag{4.5}$$

in the continuum limit (4.1) tends to $Q_t^{(0)}(x' \mid x)$ where, defining $\theta_3(z;q)$ as in Whittaker and Watson [11],

$$Q_t^{(0)}(x' \mid x) = \frac{\mathrm{e}^{-\sigma t}}{L}\theta_3(\pi(x-x')/L; \mathrm{e}^{-2\pi^2 Dt/L^2}). \tag{4.6}$$

The quantity $Q_t^{(0)}(x' \mid x)$ can be characterised as the unique solution of the generalised heat equation [4]

$$\frac{\partial Q_t^{(0)}}{\partial t} = \frac{D}{2}\frac{\partial^2 Q_t^{(0)}}{\partial x^2} - \sigma Q_t^{(0)} \tag{4.7}$$

subject to periodic boundary conditions, period L, and the initial condition

$$Q_0^{(0)}(x' \mid x) = \delta_{x',x} \qquad 0 \le x, x' \le L. \tag{4.8}$$

4.1. *A model of de Gennes*

The continuum version of the partition function (2.3) was studied by de Gennes [12], long before the discrete model was considered. This is a two-dimensional model, in the X–T plane say, consisting of N strings attached along the line $T = 0$ at equal spacing v. The other end of the strings are attached along the line $T = t$ at the points $x_1, x_2, \ldots, x_N$. Each string $x(t)$ is flexible but at an energy cost $E[x(t)]$ which is directly proportional to the length of the string. Thus

$$E[x(t)] = A\int_0^t \left[1 + \left(\frac{\mathrm{d}x}{\mathrm{d}T}\right)^2\right]^{1/2} \mathrm{d}T. \tag{4.9}$$

Assuming $\mid \mathrm{d}x/\mathrm{d}T \mid << 1$, the square root can be expanded to give

$$E_{\mathrm{dG}}[x(T)] = At + \frac{A}{2}\int_0^t \left(\frac{\mathrm{d}x}{\mathrm{d}T}\right)^2 \mathrm{d}T. \tag{4.10}$$

There are no forces between different strings except that they are not allowed to overlap. The total energy of the system is therefore the sum of self-energies of the N

strings subject to that constraint. The partition function can be written as the path integral

$$G(x'_1,\dots,x'_N \mid x_1,\dots,x_N;t) = \mathrm{e}^{-\sigma N t} \int \prod_{j=1}^{N} \exp\left[-\frac{1}{2D}\int_0^t \left(\frac{\mathrm{d}x_j}{\mathrm{d}T}\right)^2 \mathrm{d}T\right] \mathscr{D}x_j \tag{4.11}$$

where, to make contact with the results of subsection 4.1 above, βA has been denoted by σ in the factor outside the integral and by $1/D$ within the integral. The integral is over all non-intersecting paths from $x'_1,\dots,x'_N$ at $T=0$ to $x_1,\dots,x_N$ at $T=t$.

Writing $t=\mathrm{i}s$, and supposing $x_j(\mathrm{i}s)=x_j(s)$, shows that G is the path integral (with $\hbar=1$) for N fermions of mass $1/D$ in a constant potential $V(x)=\sigma$. Thus, as detailed in [13], G satisfies the Schrödinger equation

$$-\frac{D}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}G+\sigma G=\mathrm{i}\frac{\partial}{\partial s}G. \tag{4.12}$$

Since $s=-\mathrm{i}t$, we see that this equation is the N-dimensional generalisation of equation (4.7). Equation (4.12) is to be solved subject to periodic boundary conditions (period L), the initial condition

$$G(x'_1,\dots,x'_N \mid x_1,\dots,x_N;t) = \prod_{j=1}^{N}\prod_{k=1}^{N}\delta_{x'_j,x_{j+k}} \tag{4.13}$$

(with $x_j := x_{j+N}$) and the no-intersections condition

$$G(x'_1,\dots,x'_N \mid x_1,\dots,x_j,\dots,x_k,\dots,x_N;t)=0 \qquad \text{if } x_j=x_k. \tag{4.14}$$

Using (4.7), it can be readily verified that for N odd the required solution is

$$G(x'_1,\dots,x'_N \mid x_1,\dots,x_N;t) = \det\left[Q_t^{(0)}(x'_j \mid x_k)\right]_{j,k=1,\dots,N} \tag{4.15}$$

where $Q_t^{(0)}(x' \mid x)$ is given by (4.6). The right-hand side of (4.15) is precisely the continuum limit (4.4) of the discrete partition function (2.10). From this fact all quantities calculated for the discrete model can be obtained for the continuum model by taking the limit (4.4).

4.2. *A theta function identity*

We will conclude by noting that for $x'_j = Nj/L$, $j=1,\dots,N$, the determinant (4.15) can be evaluated explicitly.

Theorem 4.1. Let N be odd. Then

$$\det[\theta_3(\pi(x_j-\ell/N);q^{1/N})]_{j,\ell=1,\dots,N} = \theta_3\Big(\pi\sum_{j=1}^{N}x_j;q\Big)f_N(q)\prod_{1\le j<k\le N}\theta_1(\pi(x_k-x_j);q) \tag{4.16}$$

where

$$f_N(q) = N^{N/2} q^{-(N-1)(N-2)/24} \left\{ \prod_{k=1}^{\infty} (1 - q^{2k}) \right\}^{-(N-1)(N-2)/2} \tag{4.17}$$

This result can be proved by first using Liouville's theorem as detailed in theorem 2.1 of [14], which establishes that, up to the factor $f_N(q)$, both sides are identical. To evaluate $f_N(q)$, replace x_j by $x_j + \gamma$ for each $j = 1, \ldots, N$ in (4.16) and integrate over γ. This gives

$$\int_0^1 \mathrm{d}\gamma \det[\theta_3(\pi(x_j - \ell/N); q^{1/N})]_{j,\ell=1,\ldots,N} = f_N(q) \prod_{1 \le j < k \le N} \theta_1(\pi(x_k - x_j); q). \tag{4.18}$$

In theorem 2.3 of [14] this identity has been proved with $f_N(q)$ given by (4.17).

As well as occurring in the vicious walker problem, determinants of the form in (4.16) also occur in the calculation of correlation functions in the two-dimensional Ising model [15], and in the statistical mechanics of two-dimensional classical charges in periodic boundary conditions [14].

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